A tribute to A.K. Erlang

BY ARNE MYSKJA

In this special *teletraffic* issue of the journal Telektronikk most attention naturally focuses on the state of the art problems. Those problems are as sophisticated and as diverse as the technologies and the applications themselves. Still the editor deems it highly appropriate to throw a glance back at the roots of the theories that have developed so impressively during close to ninety years. It all began in the days of transition from manual to automatic telephony, with extremely simple network structures, one type of service and a very limited number of users. Still, the basic concepts that were identified at the time are valid even today.

The Danish scientist and telephone laboratory manager Agner Krarup Erlang (1878 – 1929) has been called "the father of teletraffic theory". Among several works on statistics as well as electrotechnical matters, most of them related to his profession within the telephone business, one work published in 1917 is generally recognised as his most important contribution. That paper contains among other things the famous B-formula. The publication and the formula are explicitly mentioned in the original proposal of 1943 that later led to the CCIF decision of 1946 to adopt "Erlang"[1] as the unit of traffic intensity.

In 1948 the Danish Academy of Technical Sciences published in the Academy's Transaction series "The Life and Works of A. K. Erlang" [1] by authors E. Brockmeyer, H.L. Halstrøm and Arne Jensen. The publication was a highly laudable initiative, as the book contains a brief biography along with all the essential written works by Erlang, as well as elucidating articles extending on Erlang's theories in an updated mathematical form. Also, explanatory comments on each one of the included papers are offered. A second edition reprint appeared in 1960 in Acta Polytechnica Scandinavica.

Most of Erlang's publications were originally written in Danish and published in Danish journals of mathematics, physics or electrotechniques. Many were later published in foreign journals in English, French and/or German language. In [1] they all appear in English translation. Also a previously unpublished work is included.

Apart from a few mathematics studies and some mathematical tables, most of Erlang's works fall into two main groups: stochastic processes, with application to telephone traffic, and electrotechniques, mainly with application to signal transmission and measurements. Within both areas he utilises mathematics as the main tool. His works within the area of stochastic processes have had the most profound influence on later developments.

The first important publication appeared in *Nyt Tidsskrift for Matematik* in 1909, where he applies basic probability theory to the case of random calls, showing that the number of calls arriving within a given length of time is given by a Poisson distribution. Also, waiting time is treated in an initial way.

The reason for mentioning the 1909 paper in particular here is threefold: firstly, it indicates the beginning of the main work that Erlang carried out later; secondly, Erlang here (as also frequently later) relates his work to previous attempts at using probability theory as a tool in telephone traffic, with a special tribute to F. Johannsen; and thirdly, he makes the introductory statement that "... a special knowledge of telephonic problems is not at all necessary for the understanding [of the theory] ... ", this statement indicating the more general applicability of the theory.

In the eight years following his 1909 publication, Erlang seems to have been preoccupied with his assignment as a leader of the laboratory, as he mainly came out with electrotechnical work and presentation of numerical tables.

The already mentioned principal paper by Erlang , first published 1917 in Danish in *Elektroteknikeren,* later appeared in British, German and French journals. In ref. [1] the paper appears in English translation: *Solution of some Problems in the Theory of Probabilities of Significance in Automatic Telephone Exchanges*. In this paper the two main formulas connected with Erlang's name both appear, the blocking formula (B-formula) and the delay formula (D-formula).

Erlang's initial assumption in mathematical modelling of blocking and delay is the more difficult proposition of constant rather than exponential holding times. He arrives at the results that the assumption is of no consequence for the blocking probability, whereas for waiting time the exponential distribution means a great simplification, permitting the simple form of the D-formula for the waiting probability. For this case even the waiting time *distribution* in its general form is given without proof, along with mean waiting time. An initial discussion of gradings is also included, as a prelude to the later (1920) presentation of his interconnection formula.

Erlang applied with explicit mention the concept of "statistical equilibrium", and in later presentations he used the graphical tool of state transition diagrams. It is clear from Erlang's works that he was fully aware of the significance of distributions, with consequences far beyond mean values. In particular, he studied holding time distributions, with focus on exponential and constant time intervals. He treated the serial addition of exponential times, leading to the distribution that also carries his name (the Erlang-k distribution), and which is a general form, with constant and exponential distributions as limit cases.

In retrospect the scientific work of A. K. Erlang is quite impressive as a pioneering work, considering his understanding of the basic concepts, and his ability to formulate in concise mathematical terms the essential properties of telephone traffic in a way that opened the road of analysis and system dimensioning that have later proved so successful. Certainly, efforts in that direction had already been initiated by others, and simultaneous embryonic work was under way by a handful of other people. However, nobody challenges the unique position of A. K. Erlang on this arena.

In tribute to Erlang's pioneering work it is appropriate to include in the present teletraffic issue of Telektronikk a reprint from reference [1] of the 1917 publication.

Reference

1. Brockmeyer, E, Halstrøm, H L, Jensen, A. The life and works of A.K. Erlang. *Transactions of the Danish Academy of Mathematical Sciences,* 2, 1948.

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[1] *The use of upper and lower case (Erlang and erlang) is not consistent. In the primary reference [1] for the present article (p. 21) the quote from the CCIF decision uses "erlang", whereas in other context in the same reference "Erlang" is applied. Similarly, in the CCITT Blue Book, Fascicle I.3, the keyword is "erlang" (also all other keywords apply lower case), whereas in the definition of traffic unit "Erlang" is applied, with symbol E.*

Solution of some problems in the theory of probabilities of significance in automatic telephone exchanges*

BY A K ERLANG

Summary. – Sections 1–7. First main problem: Systems without waiting arrangements. (Two different presuppositions.) Accompanied by Tables 1, 2, 3. Sections 8–9. Second main problem: systems with waiting arrangement (Two different presuppositions.) Accompanied by Tables 4, 5, 6, 7. Sections 10–12. Approximative methods, references, conclusion. Accompanied by Table 8.

1. First Main Problem. – Let us suppose that an automatic system is arranged in such a manner that there are provided *x* lines to take a certain number of subscribers. These *x* lines are said to be co-operative, or to constitute a "group" (or "team"). It is presupposed that all the lines disengaged are accessible. At present we will only speak of systems without waiting arrangements, i.e. systems in which the subscriber, when he finds that all *x* lines are engaged, replaces the receiver, and does not try to get connection again immediately. The probability of thus finding the lines engaged is called the loss, or degree of hindrance, and is here designated by *B*. With respect to the length of the conversations (sometimes called the holding-time), we will (for the present) suppose that it is constant, and it will be convenient to consider this quantity equal to 1 ("the natural time-unit"). With respect to the subscribers' calls, it is assumed that they are distributed quite accidentally throughout the time in question (e.g. that part of the day when the heaviest traffic usually occurs). This presupposition does not only imply that there must not be points of time within the period of time in consideration at which it may be expected in advance that there will be exceptionally many or few calls, but also that the calls must be mutually independent. In practice these presuppositions will, with great approximation, be fulfilled. The average number of calls per time-unit (intensity of traffic) is called *y*. The ratio of *y* to *x*, i.e. the traffic intensity per line, is designed by α; it is often called the efficiency of the group. We have to determine *B* (as a function of *y* and *x*). The exact expression for this is as follows:

$$B = \frac{\frac{y^x}{x!}}{1 + \frac{y}{1!} + \frac{y^2}{2!} + \ldots + \frac{y^x}{x!}} \tag{1}$$

as proved in the following sections (2–5).

2. The following proof may be characterised as belonging to the mathematical statistics, and is founded on the theory of "statistical equilibrium" – a conception which is of great value in solving certain classes of problems in the theory of probabilities. Let us consider a very great number of simultaneously operating groups of lines of the previously described kind (number of lines *x*, traffic intensity *y*). If we examine a separate group at a definite moment, we may describe its momentary condition by stating, firstly, how many of the *x* lines (0, 1, 2, ... *x*) are engaged; and secondly, how much there is left of each of the conversations in question. If we examine the same group a short time *dt* later, we will find that certain changes of two different kinds have taken place. On the one hand, the conversations which were nearly finished will now be over, and the others have become a little older. On the other hand, new calls may have been made, which, however, will have significance only if not all the lines are engaged. (The probability of a new call during the short time *dt* is *ydt*.) We assume that we examine in this manner not only one group, but a very great number of groups, both with respect to the momentary condition and the manner in which this alters. The state, of which we thus can get an accurate description, if we use a sufficiently large material, has the characteristic property that, notwithstanding the aforesaid individual alterations, it maintains itself, and, when once begun, remains unaltered, since the alterations of the different kinds balance each other. This property is called "statistic equilibrium".

3. Temporarily as a postulate, we will now set forth the following description of the state of statistical equilibrium.

The probabilities that 0, 1, 2, 3, ... *x* lines are engaged are respectively –

$$S_0 = \frac{1}{1 + \frac{y}{1!} + \frac{y^2}{2!} + \ldots + \frac{y^x}{x!}}$$

$$S_1 = \frac{\frac{y}{1}}{1 + \frac{y}{1!} + \frac{y^2}{2!} + \ldots + \frac{y^x}{x!}}$$

$$S_2 = \frac{\frac{y^2}{2}}{1 + \frac{y}{1!} + \frac{y^2}{2!} + \ldots + \frac{y^x}{x!}} \tag{2}$$

....

....

$$S_x = \frac{\frac{y^x}{x!}}{1 + \frac{y}{1!} + \frac{y^2}{2!} + \ldots + \frac{y^x}{x!}}$$

where the sum of all the probabilities is 1, as it should be. And we further postulate for each of the *x* + 1 aforesaid special conditions, that the still remaining parts of the current conversations ("remainders") will vary quite accidentally between the limits 0 and 1, so that no special value or combination of values is more probable than the others.

4. We shall prove that the thus described general state is in statistical equilibrium. For that purpose we must keep account of the fluctuations (increase and decrease), during the time *dt*, for the *x* + 1 different states, beginning with the first two. The transition from the first state S_0 to the second state S_1 amounts to

$$S_0 y dt,$$

while the transition from the second S_1 to the first S_0 amounts to

$$S_1 \cdot dt.$$

These quantities are according to (3) equal and thus cancel each other. Furthermore, the amount of transition from S_1 to S_2 is:

$$S_1 \cdot ydt,$$

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* *First published in "Elektroteknikeren" vol. 13 (1917) p.5.*

Table 1.
Values of the Loss, or Grade of Service, B. (Formula (1), Section 1).

x	α	y	B
1	0.1	0.1	0.091
1	0.2	0.2	0.167
2	0.1	0.2	0.016
2	0.2	0.4	0.054
2	0.3	0.6	0.101
3	0.1	0.3	0.003
3	0.2	0.6	0.020
3	0.3	0.9	0.050
3	0.4	1.2	0.090
4	0.1	0.4	0.001
4	0.2	0.8	0.008
4	0.3	1.2	0.026
4	0.4	1.6	0.056
5	0.2	1.0	0.003
5	0.3	1.5	0.014
5	0.4	2.0	0.037
5	0.5	2.5	0.070
6	0.2	1.2	0.001
6	0.3	1.8	0.008
6	0.4	2.4	0.024
6	0.5	3.0	0.052
8	0.3	2.4	0.002
8	0.4	3.2	0.011
8	0.5	4.0	0.030
10	0.3	3	0.001
10	0.4	4	0.005
10	0.5	5	0.018
10	0.6	6	0.043
10	0.7	7	0.079
20	0.4	8	0.000
20	0.5	10	0.002
20	0.6	12	0.010
20	0.7	14	0.079
30	0.5	15	0.000
30	0.6	18	0.003
30	0.7	21	0.014
40	0.5	20	0.000
40	0.6	24	0.001
40	0.7	28	0.007

Table 2.
Values of the intensity of traffic, y, as a function of the number of lines, x, for a loss of 1, 2, 3, 4 ‰.

x	1 ‰	2 ‰	3 ‰	4 ‰
1	0.001	0.002	0.003	0.004
2	0.046	0.065	0.081	0.094
3	0.19	0.25	0.29	0.32
4	0.44	0.53	0.60	0.66
5	0.76	0.90	0.99	1.07
6	1.15	1.33	1.45	1.54
7	1.58	1.80	1.95	2.06
8	2.05	2.31	2.48	2.62
9	2.56	2.85	3.05	3.21
10	3.09	3.43	3.65	3.82
11	3.65	4.02	4.26	4.45
12	4.23	4.64	4.90	5.11
13	4.83	5.27	5.56	5.78
14	5.45	5.92	6.23	6.47
15	6.08	6.58	6.91	7.17
16	6.72	7.26	7.61	7.88
17	6.38	7.95	8.32	8.60
18	8.05	8.64	9.03	9.33
19	8.72	9.35	9.76	10.07
20	9.41	10.07	10.50	10.82
25	12.97	13.76	14.28	14.67
30	16.68	17.61	18.20	18.66
35	20.52	21.56	22.23	22.75
40	24.44	25.6	26.3	26.9
45	28.45	29.7	30.5	31.1
50	32.5	33.9	34.8	35.4
55	36.6	38.1	39.0	39.8
60	40.8	42.3	43.4	44.1
65	45.0	46.6	47.7	48.5
70	49.2	51.0	52.1	53.0
75	53.5	55.3	56.5	57.4
80	57.8	59.7	61.0	61.9
85	62.1	64.1	65.4	66.4
90	66.5	68.6	69.9	70.9
95	70.8	73.0	74.4	75.4
100	75.2	77.5	78.9	80.0
105	79.6	82.0	83.4	84.6
110	84.1	86.4	88.0	89.2
115	88.5	91.0	92.5	93.7
120	93.0	95.5	97.1	98.4

and, conversely, the transition from S_2 to S_1 is:

$$S_2 \cdot 2 \cdot dt,$$

which two quantities also are equal and cancel each other.

Finally, we have

$$S_{x\text{-}1} \cdot y \cdot dt$$

and

$$S_x \cdot x \cdot dt,$$

which also cancel each other. The result is that the reciprocal changes which take place between the $x + 1$ different states during the time dt, compensate each other, so that the distribution remains unaltered. We still have to prove that neither will there be any alterations in the distribution of the magnitude of the remainders, i.e. that the decrease and increase, also in this respect, compensate each other.

5. Let us consider the cases in which the number of current conversations is n, and among these cases, more especially those in which the magnitudes of the n remain-

Table 3 a.
The “Loss” (in ‰ by 3 different arrangements (one with “Grading and Interconnecting”).

y	3	4	5	6	7	8	9	10	11	12
1) $x = 10$, with 10 contacts	0.8	5.3	18.4	43.1	–	–	–	–	–	–
1) $x = 18$, with 18 contacts	–	–	–	–	0.2	0.9	2.9	7.1	14.8	26.5
3) $x = 18$, with 10 contacts	–	–	–	–	1.1	3.1	7.4	15.1	26.8	42.8

Table 3 b.
Values of a and y by different arrangements for a loss of 1 ‰.

	α	y
$x = 10$; 10 contacts	0.31	3.1
$x = 18$; 10 -	0.38	6.9
$x = \infty$; 10 -	0.50	–

ders lie, respectively, between the following limits:

t_1 and $t_1 + \Delta_1$,
t_2 and $t_2 + \Delta_2$,
.. ..
.. ..
t_n and $t_n + \Delta_n$.

The probability of this is (according to Section 3):

$$\Delta_1 \cdot \Delta_2 \cdot \Delta_3 \ldots \Delta_n \cdot S_n.$$

During the time dt there may occur, in four different ways, both increase and decrease.

Firstly, transition to S_{n+1}; namely, if a call arrives; the probability of this will be:

$$\Delta_1 \cdot \Delta_2 \cdot \Delta_3 \ldots \Delta_n \cdot S_n \cdot y \cdot dt.$$

Secondly, transition from S_{n+1}; namely, if one among the $n + 1$ current conversations finishes during the time dt, and, thereafter, the n remainders lie between the above settled limits. The corresponding probability is:

$$\Delta_1 \cdot \Delta_2 \cdot \Delta_3 \ldots \Delta_n \, (n + 1) \, S_{n+1} \cdot dt,$$

which is equal to the preceding one.

Thirdly, transition from S_n itself; namely, if, among the n remainders, the $n - 1$ lie between the settled limits, and the one lies just below the lower limit in question, at a distance shorter than dt. The probability for this will be:

$$\Delta_1 \cdot \Delta_2 \cdot \Delta_3 ... \Delta_n \left(\frac{1}{\Delta_1} + \frac{1}{\Delta_2} + \ldots + \frac{1}{\Delta_n}\right) S_n \cdot dt$$

Fourthly, transition to S_n itself; namely, if, among the n remainders, the $n - 1$ lie between the settled limits, and the one lies just below the upper limit, at a distance shorter than dt. The probability of this eventuality is obviously equal to the preceding one.

Thus, there is a balance. So it is proved by this that there will be statistical equilibrium. On the other hand, any other supposition than the one set forth in Section 3 will at once be seen to be inconsistent with statistic equilibrium. The formulae in Section 3 are now proved, and thereby the proposition in Section 1 is also proved.

6. The above presupposition, that all conversations are of equal length, applies with great approximation to trunk-line conversations, but not, of course, to the usual local conversations. Now, a statistic investigation, which I have undertaken, shows that the duration of these conversations is ruled by a simple law of distribution, which may be expressed as follows:

The probability that the duration will exceed a certain time n is equal to

$$e^{-n},$$

when the average duration is taken to be equal to 1, as before. Or, in other words, the probability that a conversation which has been proceeding for some time is nearly finished, is quite independent of the length of the time which has already elapsed. The average number of conversations finished during the time dt (per current conversation) will be equal to dt. It is now easy to see that we must arrive at the same expression (1) for B as under the former presupposition, only that the proof becomes somewhat simpler, because it is necessary to take into account only the number of current conversations without paying any attention to their age. (It will appear from the following that the two aforesaid presuppositions do not lead to the same result in *all* problems.)

7. In Table 1 are shown some numerical values of the “loss” B as dependent of x and y (or α), and as given by the proposed theory.

In Table 2 the results of formula (1) are presented in another form, which is probably the one that is most useful in practice; x and B are here entry numbers, and the table gives y as a function of x and B.

In Table 3a only the first and second lines threat of systems with “pure” groups (to which formula (1) applies). The values given in the third line correspond to a different system, in which a special arrangement, the so-called “grading and interconnecting”, is used. We may describe this arrangement as follows:

The number of contacts of the selectors (here ten) is less than the number of lines (here eighteen) in the “group”. Thus each call searches not all eighteen but only ten lines. It is hereby presupposed (for the sake of simplicity) that the ten lines are each time accidentally chosen, out of the eighteen, and that they are tested one after the other according to an arbitrary selection. The method of calculation here to be used may be considered as a natural extension of the method which leads to formula (1), but it is, of course, a little more complicated. A few results of this kind of calculating are given in the two Tables 3a and 3b. Finally, I want to point out that the systems for “grading and interconnecting” being used in practice at present, which I, however, do not know in detail, are said to deviate a little from the description given here, and, therefore, it may be expected that they will give somewhat less favourable results.

8. Second Main Problem. – The problem to be considered now concerns systems

Table 4. (x = 1).

n ***a***	**0.0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**1.0**
0.00	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.05	0.950	0.955	0.960	0.964	0.969	0.974	0.979	0.984	0.989	0.994	0.999
0.10	0.900	0.909	0.918	0.927	0.937	0.946	0.956	0.965	0.975	0.985	0.995
0.15	0.850	0.863	0.876	0.889	0.903	0.916	0.930	0.944	0.958	0.973	0.988
0.20	0.800	0.816	0.833	0.849	0.867	0.884	0.902	0.920	0.939	0.958	0.977
0.25	0.750	0.769	0.788	0.808	0.829	0.850	0.871	0.893	0.916	0.939	0.963
0.30	0.700	0.721	0.743	0.766	0.789	0.814	0.838	0.864	0.890	0.916	0.946
0.35	0.650	0.673	0.697	0.722	0.748	0.774	0.802	0.830	0.860	0.891	0.922
0.40	0.600	0.624	0.650	0.677	0.704	0.733	0.763	0.794	0.826	0.860	0.895
0.45	0.550	0.575	0.602	0.630	0.658	0.689	0.720	0.754	0.788	0.825	0.863
0.50	0.500	0.526	0.553	0.581	0.611	0.642	0.675	0.710	0.746	0.784	0.824
0.55	0.450	0.475	0.502	0.531	0.561	0.592	0.626	0.661	0.699	0.738	0.780
0.60	0.400	0.425	0.451	0.479	0.508	0.540	0.573	0.609	0.646	0.686	0.729
0.65	0.350	0.374	0.399	0.425	0.454	0.484	0.517	0.552	0.589	0.628	0.670
0.70	0.300	0.322	0.345	0.370	0.397	0.426	0.457	0.490	0.525	0.563	0.604
0.75	0.250	0.269	0.290	0.313	0.337	0.364	0.392	0.423	0.456	0.491	0.529
0.80	0.200	0.217	0.235	0.254	0.275	0.298	0.323	0.350	0.379	0.411	0.445
0.85	0.150	0.163	0.178	0.194	0.211	0.339	0.250	0.272	0.296	0.322	0.351
0.90	0.100	0.109	0.120	0.131	0.143	0.157	0.172	0.188	0.205	0.225	0.246
0.95	0.050	0.055	0.060	0.066	0.073	0.080	0.088	0.097	0.107	0.118	0.129
1.00	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Table 5. (x = 2).

n ***a***	**0.0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**1.0**
0.00	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.05	0.995	0.996	0.997	0.998	0.998	0.999	0.999	1.000	1.000	1.000	1.000
0.10	0.982	0.985	0.988	0.991	0.993	0.995	0.997	0.998	0.999	1.000	1.000
0.15	0.962	0.968	0.974	0.980	0.985	0.989	0.993	0.996	0.998	0.999	1.000
0.20	0.936	0.946	0.956	0.964	0.973	0.980	0.987	0.992	0.995	0.998	0.999
0.25	0.903	0.918	0.933	0.946	0.958	0.969	0.978	0.986	0.992	0.996	0.998
0.30	0.866	0.886	0.905	0.922	0.939	0.953	0.967	0.978	0.986	0.992	0.995
0.35	0.825	0.849	0.872	0.895	0.915	0.935	0.952	0.966	0.978	0.987	0.991
0.40	0.779	0.808	0.835	0.862	0.888	0.911	0.933	0.952	0.967	0.978	0.985
0.45	0.730	0.762	0.794	0.825	0.855	0.883	0.909	0.932	0.951	0.966	0.975
0.50	0.677	0.712	0.748	0.783	0.817	0.849	0.880	0.907	0.931	0.949	0.961
0.55	0.621	0.658	0.697	0.735	0.773	0.809	0.844	0.875	0.903	0.925	0.941
0.60	0.561	0.601	0.641	0.681	0.722	0.762	0.800	0.836	0.868	0.894	0.913
0.65	0.499	0.539	0.580	0.622	0.664	0.706	0.748	0.787	0.822	0.852	0.875
0.70	0.435	0.473	0.514	0.556	0.599	0.642	0.685	0.726	0.764	0.798	0.824
0.75	0.368	0.404	0.442	0.483	0.525	0.568	0.611	0.653	0.692	0.728	0.757
0.80	0.298	0.331	0.366	0.403	0.442	0.482	0.523	0.564	0.603	0.639	0.669
0.85	0.227	0.254	0.283	0.315	0.348	0.384	0.420	0.457	0.493	0.527	0.556
0.90	0.153	0.173	0.195	0.219	0.244	0.272	0.300	0.330	0.359	0.387	0.412
0.95	0.077	0.088	0.101	0.114	0.129	0.144	0.161	0.179	0.196	0.214	0.230
1.00	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Table 6. ($x = 3$).

n *a*	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.00	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.05	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.10	0.996	0.997	0.998	0.999	0.999	1.000	1.000	1.000	1.000	1.000	1.000
0.15	0.989	0.992	0.994	0.996	0.997	0.998	0.999	1.000	1.000	1.000	1.000
0.20	0.976	0.982	0.987	0.991	0.994	0.996	0.998	0.999	1.000	1.000	1.000
0.25	0.958	0.967	0.975	0.983	0.988	0.993	0.996	0.998	0.999	1.000	1.000
0.30	0.933	0.948	0.960	0.971	0.980	0.987	0.992	0.996	0.998	0.999	0.999
0.35	0.903	0.923	0.940	0.956	0.969	0.979	0.987	0.993	0.996	0.998	0.999
0.40	0.866	0.892	0.915	0.936	0.953	0.968	0.980	0.988	0.993	0.996	0.998
0.45	0.823	0.855	0.884	0.910	0.934	0.953	0.969	0.980	0.988	0.993	0.995
0.50	0.775	0.812	0.847	0.879	0.908	0.933	0.953	0.969	0.980	0.987	0.991
0.55	0.720	0.762	0.803	0.841	0.876	0.906	0.932	0.952	0.967	0.977	0.983
0.60	0.660	0.706	0.752	0.795	0.835	0.872	0.903	0.929	0.948	0.962	0.971
0.65	0.595	0.644	0.693	0.740	0.786	0.827	0.864	0.895	0.920	0.938	0.951
0.70	0.524	0.574	0.625	0.676	0.725	0.771	0.813	0.849	0.879	0.902	0.919
0.75	0.448	0.496	0.548	0.600	0.651	0.700	0.746	0.787	0.821	0.849	0.871
0.80	0.367	0.413	0.461	0.511	0.562	0.611	0.659	0.702	0.740	0.773	0.799
0.85	0.282	0.322	0.364	0.409	0.455	0.501	0.547	0.590	0.629	0.663	0.693
0.90	0.192	0.222	0.255	0.291	0.328	0.366	0.405	0.442	0.477	0.509	0.538
0.95	0.098	0.115	0.134	0.155	0.177	0.201	0.225	0.249	0.273	0.295	0.316
1.00	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

with waiting arrangements. Here, the problem to be solved is determining the probability $S\ (> n)$ of a waiting time greater than an arbitrary number n, greater than or equal to zero. The last case is the one which is most frequently asked for. In the same manner we define $S\ (< n)$ where $S\ (< n) + S\ (> n) = 1$. Furthermore, we may ask about the average waiting time M. We shall answer these questions in the following. Here, too, we may begin by assuming that the duration of the conversations is constant and equal to 1. The accurate treatment of this case gives rise to rather difficult calculations, which, however, are unavoidable. Among other things, we find that we cannot use the same formula for $S\ (> n)$ for all values of n, but we must distinguish between the various successive "periods", or spaces of time of the length l. In practice, however, the first period will, as a rule, be the most important. I shall content myself by giving, without proof, the necessary formulae for the cases of $x = 1$, 2, and 3, and then (chiefly for the purpose of showing the possibility of carrying out the practical calculation) the corresponding numerical results, also for $x = 1, 2, 3$. Formulae and results for $x = 1$ have already been published in an article in "Nyt Tidsskrift for Mathematik", B, 20, 1909. The formulae for greater values of x, e.g. $x = 10$, $x = 20$ are quite analogous to those given here.

Collection of formulae

Presupposition: the duration of conversations is constant and equal to l

Denotations:

x is the number of co-operating lines

y is the intensity of traffic (average number of calls during unit of time)

$$\frac{y}{x} = \alpha$$

$S\ (> n)$ is the probability of a waiting time greater than n

$S\ (< n)$ is the probability of a waiting time less than, or equal to n

$ny = z$

$z - y = u$

$z - 2y = v$, et cetera.

M = the average waiting time.

I. Formulae for the case of $x = 1$:

a) First period, $0 < n < 1$:
$S\ (< n) = a_0 \cdot e^z$,
where $a_0 = 1 - a$

b) Second period, $1 < n < 2$:
$S\ (< n) = (b_1 - b_0 u)e^u$,
where $b_1 = a_0 e^y$
$b_0 = a_0$

c) Third period, $2 < n < 3$:
$S\ (< n) = (c_2 - c_1 v + {}^1\!/_2\, c_0 v^2)e^v$
where $c_2 = (b_1 - b_0 y)e^y$
$c_1 = b_1$
$c_0 = b_0$
et cetera.

$$M = \frac{1}{y}\left((1 - b_1) + (1 - c_2) + (1 - d_3) + \ldots\right)$$

$$= \frac{1}{2} \cdot \frac{\alpha}{1 - \alpha}$$

II. Formulae for the case of $x = 2$:

a) First period, $0 < n < 1$:
$S\ (< n) = (a_1 - a_0 z)e^z$
where

$$a_1 = 2(1 - \alpha)\frac{\alpha}{\alpha - \beta},$$

$$a_0 = -2(1 - \alpha)\frac{\beta}{\alpha - \beta}$$

β denoting the negative root of the equation

$$\beta e^{-\beta} = -\alpha e^{-\alpha}.$$

b) Second period, $1 < n < 2$:

$$S(< n) = (b_3 - b_2 u + {}^1/_2\, b_1 u^2 - {}^1/_6\, b_0 u^3) e^u$$

where

$$b_3 = (a_1 - a_0 y) e^y$$
$$b^2 = a_0 e^y$$
$$b_1 = a_1$$
$$b_0 = a_0$$

c) Third period, $2 < n < 3$:

$$S(< n) = (c_5 - c_4 v + {}^1/_2\, c_3 v^2 - {}^1/_6\, c_2 v^3 + {}^1/_{24}\, c_1 v^4 - {}^1/_{120}\, c_0 v^5) e^v$$

where

$$c_5 = (b_3 - b_2 y + {}^1/_2\, b_1 y^2 - {}^1/_6\, b_0 y^3) e^y$$
$$c_4 = (b_2 - b_1 y + {}^1/_2\, b_0 y^2) e^y$$
$$c_3 = b_3$$
$$c_2 = b_2$$
$$c_1 = b_1$$
$$c_0 = b_0$$

et cetera.

$$M = \frac{1}{y}((1 - b_2) + (1 - b_3) + (1 - c_4)$$
$$+ (1 - c_5) + (1 - d_6) + (1 - d_7) + ...)$$

III. Formulae for the case of $x = 3$

a) First period, $0 < n < 1$:

$$S\,(< n) = (a_2 - a_1 z + {}^1/_2\, a_0 z^2) e^z$$

where

$$a_2 = 3(1 - \alpha)\frac{\alpha^2}{(\alpha - \beta)(\alpha - \gamma)}$$

$$a_1 = -3(1 - \alpha)\frac{\alpha(\beta + \gamma)}{(\alpha - \beta)(\alpha - \gamma)}$$

$$a_0 = 3(1 - \alpha)\frac{\beta \cdot \gamma}{(\alpha - \beta)(\alpha - \gamma)}$$

as $\beta \cdot e^{-\beta} = \alpha \cdot e^{-\alpha} \cdot k$

$\gamma \cdot e^{-\gamma} = \alpha \cdot e^{-\alpha} \cdot k^2$

We understand by k a complex value of $\sqrt[3]{1}$

b) Second period, $1 < n < 2$:

$$S\,(< n) = (b_5 - b_4 u + {}^1/_2\, b_3 u^2 - {}^1/_6\, b_2 u^3 + {}^1/_{24}\, b_1 u^4 - {}^1/_{120}\, b_0 u^5) e^u,$$

where

$$b_5 = (a_2 - a_1 y + {}^1/_2\, a_0 y^2) e^y$$
$$b_4 = (a_1 - a_0 y) e^y$$
$$b_3 = a_0 e^y$$
$$b_2 = a_2$$
$$b_1 = a_1$$
$$b_0 = a_0$$

c) Third period, $2 < n < 3$:

$$S\,(< n) = (c_8 - c_7 v + {}^1/_2\, c_6 v^2 - {}^1/_6\, c_5 v^3 + {}^1/_{24}\, c_4 v^4 - {}^1/_{120}\, c_3 v^5 + {}^1/_{720}\, c_2 v^6 - {}^1/_{5040}\, c_1 v^7 + {}^1/_{40320}\, c_0 v^8) e^v,$$

where

$$c_8 = (b_5 - b_4 y + {}^1/_2\, b_3 y^2 - {}^1/_6\, b_2 y^3 + {}^1/_{24}\, b_1 y^4 - {}^1/_{120}\, b_0 y^5) e^y$$

$$c_7 = (b_4 - b_3 y + {}^1/_2\, b_2 y^2 - {}^1/_6\, b_1 y^3 + {}^1/_{24}\, b_0 y^4) e^y$$

$$c_6 = (b_3 - b_2 y + {}^1/_2\, b_1 y^2 - {}^1/_6\, b_0 y^3) e^y$$
$$c_5 = b_5$$
$$c_4 = b_4$$
$$c_3 = b_3$$
$$c_2 = b_2$$
$$c_1 = b_1$$
$$c_0 = b_0,$$

et cetera.

$$M = \frac{1}{7}((1 - b_3) + (1 - b_4) + (1 - b_5)$$
$$+ (1 - c_6) + (1 - c_7) + (1 - c_8) + ...)$$

9. There still remains the problem of investigating the magnitude of the waiting times in systems with waiting arrangement under the second presupposition, namely, that the durations of the conversations vary in the manner already described in Section 6.

Here we find, without difficulty, the following two formulae:

$$S\,(> 0) = c \tag{3}$$

$$S\,(> n) = c \cdot e^{-(x-y)n} \tag{4}$$

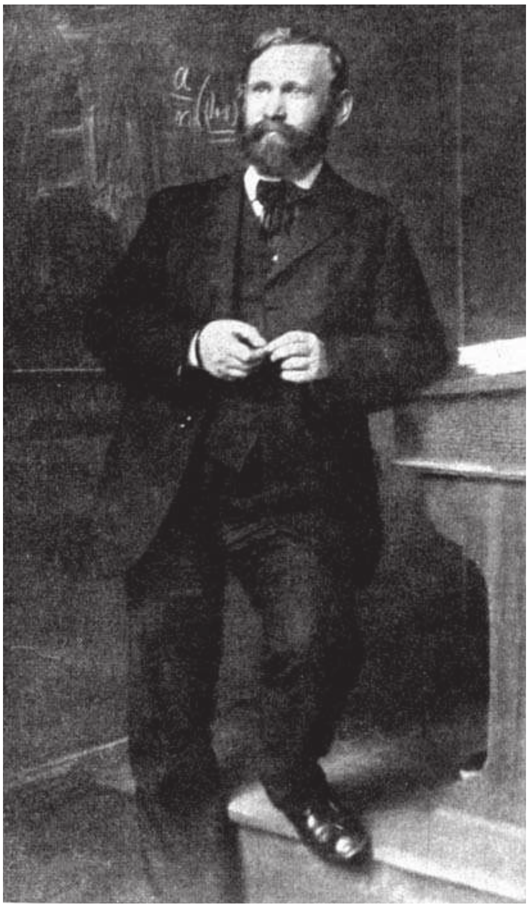
where

$$c = \frac{\frac{y^x}{x!} \cdot \frac{x}{x-y}}{1 + \frac{y}{1!} + \frac{y^2}{2!} + ... + \frac{y^{x-1}}{(x-1)!} + \frac{y^x}{x!} \cdot \frac{x}{x-y}} \tag{5}$$

while x and y have the same significance as before, and the average duration of a conversation is equal to 1. The formula is exact for all values of $n > 0$.

For the average waiting time we get the formula:

$$M = \frac{c}{x - y} \tag{6}$$



A.K. Erlang in 1910, 32 years old (from [1])

The numerical calculation causes no special difficulty. It ought, perhaps, to be pointed out that, both here and in Section 8, it is presupposed that the waiting calls are despatched in the order in which they have been received. If this does not take place in practice, it will, of course, have a slight effect upon the value of $S\,(> n)$, but not at all on the value of M, neither on $S\,(> 0)$.

10. Approximate Formulae. – The exact formulae given above are throughout so convenient, that there is hardly any need of approximative formulae. This does not, however, apply to the formulae which concern the second main problem, first presupposition. Therefore, it may be worth while to mention a couple of approximative methods which quickly lead to a serviceable result, at least in such cases as have the greatest practical significance.

One of these methods has already been used by me, at the request of Mr. *P.V. Christensen,* Assistant Chief Engineer to the Copenhagen Telephone Company, for calculating the explicit tables given in the first pages of his fundamental work, "The Number of Selectors in Automatic

Telephone Systems" (published in the Post Office Electrical Engineers' Journal, October, 1914, p. 271; also in "Elektroteknikeren", 1913, p. 207; "E.T.Z.", 1913, p. 1314).

Since the method used has not been described in full, I shall here say a few words about the same. The probability of just x calls being originated during a period of time for which the average number is y, is, as well known, under the usual presuppositions (Section 1):

$$e^{-y}\frac{y^x}{x!}$$

The mathematical theorem here used is due to *S.D. Poisson* ("Recherches sur la probabilité, etc.", 1835), and has later been studied by *L.v. Bortkewitsch* ("Das Gesetz der kleinen Zahlen", 1898). The function has been tabulated by the latter (*loc. cit.*), and later by *H.E. Soper* ("Biometrica", vol. X, 1914; also in *K. Pearson* "Tables for Statisticians, etc.", 1914).

Thus the probability of x or more calls during the mentioned period of time is:

$$P = e^{-y}\frac{y^x}{x!} + e^{-y}\frac{y^{x+1}}{(x+1)!} + e^{-y}\frac{y^{x+2}}{(x+2)!} + \dots \qquad (7)$$

It will then be seen that P, in many cases, viz. when y is not unproportionally great, will be a good approximate value for the fraction of the calls which will find all the lines engaged (or for "the probability of not getting through"). Thus P in the case of exchanges without waiting arrangements approximates the *"loss"*, and here gives obviously a somewhat too great value. In exchanges with waiting arrangement P approximates the quantity $S\ (> 0)$, the probability of delay, and gives here a somewhat too small value. Or, if it is the fraction named above which is given beforehand, as is generally the case in practice, where often the value 0.001 is used, the formula will show the connexion between y and x. The values of y found in this manner (see Table 8) will never deviate 5 per cent. from the correct values in systems without waiting arrangement; never 1 per cent. in systems with waiting arrangement (both presuppositions), if we take the named, frequently used value of P = 0.001. Possible intermediate systems between the two main classes of exchanges may, of course, be treated with good results according to the same method.

Table 7.
Systems with Waiting Arrangement (Second Presupposition). Values of S (> n) and M.

x	**α**	**y**	**S(> 0)**	**S(> 0.1)**	**S(> 0.2)**	**M**
1	0.1	0.1	0.100	0.091	0.084	0.111
1	0.2	0.2	0.200	0.185	0.170	0.250
2	0.1	0.2	0.018	0.015	0.013	0.010
2	0.2	0.4	0.067	0.057	0.049	0.042
2	0.3	0.6	0.138	0.120	0.104	0.099
3	0.1	0.3	0.004	0.003	0.002	0.001
3	0.2	0.6	0.024	0.019	0.015	0.010
3	0.3	0.9	0.070	0.057	0.046	0.033
3	0.4	1.2	0.141	0.118	0.099	0.078
4	0.1	0.4	0.001	0.001	0.000	0.000
4	0.2	0.8	0.010	0.007	0.005	0.003
4	0.3	1.2	0.037	0.028	0.022	0.013
4	0.4	1.6	0.091	0.072	0.056	0.038
5	0.2	1.0	0.004	0.003	0.002	0.001
5	0.3	1.5	0.020	0.014	0.010	0.006
5	0.4	2.0	0.060	0.044	0.033	0.020
5	0.5	2.5	0.130	0.102	0.079	0.052
6	0.2	1.2	0.002	0.001	0.001	0.000
6	0.3	1.8	0.011	0.007	0.005	0.003
6	0.4	2.4	0.040	0.026	0.018	0.011
6	0.5	3.0	0.099	0.073	0.054	0.033
8	0.3	2.4	0.004	0.002	0.001	0.001
8	0.4	3.2	0.018	0.011	0.007	0.004
8	0.5	4.0	0.059	0.040	0.026	0.015
10	0.3	3	0.001	0.001	0.000	0.000
10	0.4	4	0.009	0.005	0.003	0.001
10	0.5	5	0.036	0.022	0.013	0.007
10	0.6	6	0.102	0.068	0.046	0.026
10	0.7	7	0.222	0.165	0.122	0.074
20	0.4	8	0.000	0.000	0.000	0.000
20	0.5	10	0.008	0.001	0.001	0.000
20	0.6	12	0.024	0.011	0.005	0.003
20	0.7	14	0.094	0.052	0.028	0.016
22	0.5	11.0	0.002	0.001	0.000	0.000
22	0.6	13.2	0.018	0.007	0.003	0.002
22	0.7	15.4	0.081	0.042	0.022	0.012
30	0.5	15	0.000	0.000	0.000	0.000
30	0.6	18	0.007	0.002	0.001	0.001
30	0.7	21	0.044	0.018	0.007	0.005
40	0.5	20	0.000	0.000	0.000	0.000
40	0.6	24	0.002	0.000	0.000	0.000
40	0.7	28	0.022	0.007	0.002	0.002

If, in systems with waiting arrangement, we ask about the number of waiting times beyond a certain limit *n*, $S\ (> n)$, an extension of the same formula may be used, y being replaced by $y(1 - n)$. The method is best suited for small values of n, and the error goes to the same side as mentioned above. Furthermore, it may be mentioned in this connexion that if we use, in the case considered, the formulae following from presupposition No. 2, instead of those based upon presupposition No. 1, the errors thus introduced will be small, as a rule; they go, this time, in such a direction that we get too *great* values for $S\ (> 0)$ and $S\ (> n)$; or, if it is y which is sought, there will be too small values for y.

Table 8.
Values of y as a function of x, for P = 0.001 – 0.002 – 0.003 – 0.004.

x	1 ‰	2 ‰	3 ‰	4 ‰
1	0.001	0.002	0.003	0.004
2	0.045	0.065	0.08	0.09
3	0.19	0.24	0.28	0.31
4	0.42	0.52	0.58	0.63
5	0.73	0.86	0.95	1.02
6	1.11	1.27	1.38	1.46
7	1.52	1.72	1.85	1.95
8	1.97	2.20	2.35	2.47
9	2.45	2.72	2.89	3.02
10	2.96	3.25	3.45	3.60
11	3.49	3.82	4.03	4.19
12	4.04	4.41	4.62	4.81
13	4.61	5.00	5.24	5.43
14	5.19	5.61	5.87	6.07
15	5.79	6.23	6.51	6.72
16	6.40	6.86	7.16	7.38
17	7.03	7.51	7.82	8.06
18	7.66	8.17	8.49	8.74
19	8.31	8.84	9.18	9.44
20	8.96	9.51	9.87	10.14
21	9.61	10.20	10.57	10.84
22	10.28	10.89	11.27	11.56
23	10.96	11.59	11.98	12.28
24	11.65	12.29	12.70	13.01
25	12.34	13.00	13.42	13.74
30	15.87	16.6	17.1	17.4
35	19.5	20.4	20.9	21.3
40	23.5	24.2	24.8	25.2
45	27.1	28.1	28.7	29.2
50	30.9	32.0	32.7	33.2
55	34.9	36.0	36.8	37.3
60	38.9	40.1	40.9	41.4
65	43.0	44.2	45.0	45.6
70	47.0	48.3	49.2	49.8
75	51.0	52.4	53.3	54.0
80	55.1	56.6	57.6	58.3
85	59.3	60.9	61.8	62.5
90	63.5	65.1	66.1	66.9
95	67.7	69.3	70.4	71.1
100	71.9	73.6	74.7	75.5
105	76.2	77.9	79.0	79.8
110	80.4	82.2	83.3	84.2
115	84.7	86.6	87.7	88.5
120	89.0	90.9	92.1	93.0

11. It will be too lengthy to describe or mention, in this connexion, all the systematic practical experiments and measurements (also only partly published), which of late years have been made, partly by the firms in question (especially, Siemens and Halske, and Western Electric Co.), partly by others, or such purely empirical formulae as have thus been set forth. On the other hand, it would be incorrect to neglect one or two interesting theoretical works from recent years, which directly concern one of the problems treated above. In his doctor's thesis, Mr. *F. Spiecker* ("Die Abhängigkeit des erfolgreichen Fernsprechanrufes von der Anzahl der Verbindungsorgane", 1913), has indicated a method for determining the loss in systems without waiting arrangement, which (as he himself admits) is not quite free from errors, and which, besides, is so complicated that it can hardly find application in practice. It should be emphasized, however, that the results in the cases in which the author has completed his calculations, lie very near the results of formula (1) given above. In the same work is also given an approximative formula, which can best be compared with the formula for *P* (Section 10 above). The difference is exclusively due to a somewhat deviating, and probably less practical, formulation of the problem. Mr. *W.H. Grinsted*, in his treatise, "A Study of Telephone Traffic Problems, etc." (Post Office Electrical Engineers' Journal, April 1915), presents a solution of the same problem. Since this solution has, probably, by many readers as well as by the author himself, been considered mathematically exact, it should be noticed that an error has occurred in the derivation of the formula in question and that, for this reason, the formula gives rather incorrect results. It should be added that the treatise is a reprint of an older work from 1907 (which I have not had opportunity to examine). In spite of the faulty results, Grinsted's work is, however, when its time of publication is considered, of no little merit.

12. In closing this article, I feel called upon to render my best thanks to Mr. *F. Johannsen*, Managing Director of the Copenhagen Telephone Co., not only for his interest in the investigations recorded here, but also for his energetic initiative in starting rational and scientific treatment of many different problems in connexion with telephone traffic. I also owe many thanks to Mr. *J.L.W.V. Jensen*, Engineer-in-Chief to the same Company, for his valuable assistance especially in the treatment of some mathematical difficulties.